Characterization of mass-stationarity by Bernoulli and Cox transports

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Abstract

Consider a random measure ξ on a locally compact Abelian group G acting on some random element X. Mass-stationarity – introduced in [6] – means (informally) that the origin is a typical location for (X, ξ) in the mass of ξ. It is an intrinsic characterization of Palm versions w.r.t stationary random measures. In this paper we show that mass-stationarity w.r.t. discrete ξ is characterized by distributional invariance under shifts of the origin by certain mass-preserving transports involving a Bernoulli randomization of the group-identity and an allocation rule. We also show that mass-stationarity w.r.t. a general ξ is characterized by mass-stationarity w.r.t. a Cox process driven by ξ.

1 Introduction

Let ξ be a random measure on a locally compact Abelian group G. *Mass-stationarity* is a formalization of the intuitive idea that the origin is a typical location in the mass of ξ, just like stationarity means that the origin is a typical location in the space G. The formal definition is given in Section 2 below. Actually, we will consider ξ jointly with a random element X which G acts on, for instance a random field indexed by G. Then (X, ξ) is *mass-stationary* if the origin is a typical location for (X, ξ) in the mass of ξ.

The word 'typical' needs some explanation. If ξ is finite and S is a random element in G with conditional distribution $\xi/\xi(G)$ given ξ, then we say that S is a *typical* location in the *mass* of ξ. We also say that the *origin* is a typical location in the mass of the *shifted* measure $\xi(\cdot - S)$. Further, if S has conditional distribution $\xi/\xi(G)$ given (X, ξ), then we say that S is a *typical* location for (X, ξ) in the mass of ξ, and also that the origin is a *typical* location in the mass of $\xi(\cdot - S)$ for the pair (X, ξ) shifted by S. In this introduction we use the term 'typical' even for infinite ξ in order to explain informally the basic ideas of the paper.

Mass-stationarity was introduced in [6] as an extension to random measures of *point-stationarity*, which in turn was introduced in [8] for simple point processes in $\mathbb{R}^d$ having a point at the origin. Point-stationarity formalizes the intuitive idea that the point at the origin is a typical point of the point process (think of the Poisson process on the line with an extra point added at the origin: shifting the origin to the n^{th} point on the right – or to the n^{th} point on the left – does not change the fact that the inter-point distances are i.i.d. exponential). The definition in [8] involved an external randomization, but in [1] (and in [2] for the group case) it is shown that point-stationarity can be defined as 'distributional invariance under shifts of the origin by preserving allocation rules': an *allocation rule* τ is a map taking each location $s \in G$ to another location $\tau(s) \in G$ depending on $\xi(\cdot - s)$, and τ is *preserving* if the image of ξ under τ is ξ itself. In fact, [1] and [2] show that 'matchings' suffice for the definition: an allocation rule τ is a *matching* if τ is its own inverse.

In [8] it was shown that point-stationarity is an intrinsic characterization of Palm versions of stationary point processes, and the same is proved in [6] for mass-stationarity and random measures. In this paper we will derive further characterizations of mass-stationarity.

The term 'Bernoulli transport' refers to a randomized allocation rule that allows staying at a location s with a probability $p(s)$ depending on $\xi(\cdot - s)$, and otherwise chooses another location according to a (non-randomized) allocation rule. This makes it possible to preserve discrete point-masses even if there are point-masses of different sizes. In Section 3 we show that mass-stationarity of discrete random measures can be reduced to distributional invariance of ξ under shifts of the origin by preserving Bernoulli transports, Theorem 3.2. A similar result holds for random pairs (X, ξ).

A Cox process ζ is a Poisson process with a random intensity measure ξ. Such a process can be thought of as a collection of points scattered independently over the space G according to the mass distribution of ξ, so these points are at typical locations in the mass of ξ. Thus if ξ is mass-stationary and we add a point at the origin to the Cox process to obtain $\zeta^0 := \zeta + \delta_0$, then also the points of ζ^0 are at typical locations in the mass of ξ. In fact, one might expect that the new point at the origin is a typical point of ζ^0, in other words that ζ^0 is point-stationary, and even that the pair (ξ, ζ^0) is point-stationary. Actually, one might expect that the pair (ξ, ζ^0) is point-stationary *if and only if* ξ is mass-stationary. In Section 4 we show that this is indeed the case. In fact, the result extends to random pairs (X, ξ), Theorem 4.1.

The term 'Cox transport' refers to applying an allocation rule to a Cox process driven by a general random measure (think of the mass of the random measure being represented by the points of the Cox process). In particular, mass-stationarity of ξ then reduces to point-stationarity with respect to ζ^0, Theorem 4.1. Also, it follows that mass-stationarity is characterized by applying preserving Bernoulli transports to the Cox process, Corollary 4.3. Finally, for diffuse random measures mass-stationarity is characterized by applying matchings to the Cox process, Corollary 4.4.

2 Transports and mass-stationarity

We consider a topologial Abelian group G that is assumed to be a locally compact, second countable Hausdorff space with Borel σ-field $\mathcal{G}$ and Haar measure λ. Let M denote the set of all locally finite measures on G equipped with the cylindrical σ-field $\mathcal{M}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$

be a σ-finite measure space. Although $\mathbb{P}$ need not be a probability measure, we still use a probabilistic language. A *random measure* is a random element ξ in M. We use the kernel notation $\xi(\omega,\cdot) := \xi(\omega)(\cdot)$, $\omega \in \Omega$. We equip $(M,\mathcal{M})$ with a *measurable flow* $\theta_s : M \to M$, $s \in G$, defined by $\theta_s\mu(B) := \mu(B+s)$, where $B \in \mathcal{G}$ and $B + s := \{t + s : t \in B\}$. Then $(\mu, s) \mapsto \theta_s\mu$ is a measurable mapping, θ_0 is the identity on M, and we have the flow property

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s,t \in G. \tag{2.1}$$

Here 0 denotes the neutral element in G and $\circ$ denotes composition. Together with ξ we consider a random element X in a measurable space $(W,\mathcal{W})$. We assume that this space is equipped with a measurable flow $\theta_s : W \to W$, $s \in G$, having the properties listed above. (Denoting this flow again by θ_s, $s \in G$, will cause no risk of ambiguity.)

Next we adapt some terminology from [6] to the setting established above. This makes some of the definitions more cumbersome. However, the present setting is closer to chapter 11 of [4] and chapter 9 of [9] and will allow for a more convenient formulation of our main results in Section 4. In the remainder of this paper we consider a pair (X,ξ) as introduced above such that $\mathbb{P}((X,\xi) \in \cdot)$ is σ-finite and $\mathbb{P}(\xi(G) = 0) = 0$. We call (X,ξ) *stationary* if $\mathbb{P}(\theta_s(X,\xi) \in \cdot) = \mathbb{P}((X,\xi) \in \cdot)$ for all $s \in G$. Here we define $\theta_s(w,\mu) := (\theta_s w, \theta_s\mu)$ for $s \in G$ and $(w,\mu) \in W \times M$. If (X,ξ) is stationary, then we also call $\mathbb{P}((X,\xi) \in \cdot)$ *invariant.* In this case the measure

$$\mathbb{P}_{X,\xi}(A) := \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(X(\omega),\xi(\omega)))\mathbf{1}_B(s)\,\xi(\omega,ds)\,\mathbb{P}(d\omega), \quad A \in \mathcal{W}\otimes\mathcal{M}, \tag{2.2}$$

is called the *Palm measure* of (X,ξ) (with respect to $\mathbb{P}$), see [7]. Here $B \in \mathcal{G}$ has $0 < \lambda(B) < \infty$. This measure is σ-finite. As the definition (2.2) is independent of B, we can use a monotone class argument to conclude the *refined Campbell theorem*

$$\iint f(\theta_s(X(\omega),\xi(\omega)),s)\,\xi(\omega,ds)\,\mathbb{P}(d\omega) = \iint f(x,\mu,s)\,ds\,\mathbb{P}_{X,\xi}(d(x,\mu))$$

for all measurable $f : W \times M \times G \to [0,\infty)$, where ds refers to integration with respect to the Haar measure λ. Using a standard convention in probability theory, we write this as

$$\mathbb{E}_{\mathbb{P}}\left[\int f(\theta_s(X,\xi),s)\,\xi(ds)\right] = \mathbb{E}_{\mathbb{P}_{X,\xi}}\left[\int f(X,\xi,s)\,ds\right], \tag{2.3}$$

where $\mathbb{E}_{\mathbb{P}}$ and $\mathbb{E}_{\mathbb{P}_{X,\xi}}$ denote integration with respect to $\mathbb{P}$ and $\mathbb{P}_{X,\xi}$, respectively.

Next we define *mass-stationarity* of (X,ξ). Let $C \in \mathcal{G}$ be a relatively compact set having $\lambda(C) > 0$ and $\lambda(\partial C) = 0$, where ∂C denotes the boundary of C. Let U,V be random elements in G, possibly obtained by extending $(\Omega,\mathcal{F},\mathbb{P})$. Assume that (X,ξ) and U are independent, U has the uniform distribution on C (w.r.t. Haar measure), and that the conditional distribution of V given (X,ξ,U) is uniform in the mass of ξ on $C - U$. Then (X,ξ) is called *mass-stationary* if

$$(\theta_V(X,\xi), U+V) \stackrel{d}{=} (X,\xi,U) \tag{2.4}$$

holds for all such C. In this case we call the distribution $\mathbb{P}((X,\xi) \in \cdot)$ mass-stationary. By Theorem 6.3 in [6] this is equivalent to the validity of the Mecke equation

$$\mathbb{E}_{\mathbb{P}}\left[\int g(\theta_s(X,\xi), -s)\,\xi(ds)\right] = \mathbb{E}_{\mathbb{P}}\left[\int g(X,\xi,s)\,\xi(ds)\right] \tag{2.5}$$

for all measurable $g : W \times M \times G \to [0,\infty)$.

Remark 2.1. The random element X is *stationary* if $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$ for all $s \in G$ and if this measure is σ-finite. Mass-stationarity generalizes this concept. Indeed, assuming (2.5) for $\xi = \lambda$ we easily get that $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$ for λ-a.e. $s \in G$. Assuming that W is a metric space with Borel σ-field $\mathcal{W}$ and that $s \mapsto \theta_s X$ is $\mathbb{P}$-a.e. continuous, we obtain stationarity of X.

Remark 2.2. By definition, mass-stationarity of (X,ξ) is equivalent to mass-stationarity of $((X,\xi),\xi)$.

For the next definitions it is convenient to abbreviate $\Omega' := W \times M$ and $\mathcal{F}' := \mathcal{W} \otimes \mathcal{M}$. A *weighted transport-kernel* is a kernel T from $\Omega' \times G$ to G such that $T(\omega', s, \cdot)$ is locally finite for all $(\omega', s) \in \Omega' \times G$. If T is Markovian, then it is called *transport-kernel.* A weighted transport-kernel is *invariant* if $T(\theta_s\omega', 0, B - s) = T(\omega', s, B)$ for all $(\omega', s) \in \Omega' \times G$ and $B \in \mathcal{G}$. An *allocation rule* is a measurable mapping $\tau : \Omega' \times G \to G$ which is *covariant*, i.e. which has $\tau(\theta_s\omega', 0) = \tau(\omega', s) - s$ for all ω', s. A weighted transport-kernel T is *mass-preserving* if

$$\int T(w,\mu,s,\cdot)\,\mu(ds) = \mu(\cdot) \tag{2.6}$$

holds for all $(w,\mu) \in \Omega'$. An allocation rule is *mass-preserving* if

$$\int \mathbf{1}\{\tau(w,\mu,s) \in \cdot\}\,\mu(ds) = \mu(\cdot) \tag{2.7}$$

holds for all $(w,\mu) \in \Omega'$. If these relations hold almost everywhere w.r.t. some measure $\mathbb{Q}$ on Ω', then we say that T (resp. τ) is $\mathbb{Q}$-a.e. mass-preserving.

Remark 2.3. Let T be a locally finite kernel from $W \times M \times G$ to G. Assume that there is some $A \in \mathcal{W} \otimes \mathcal{M}$ such that

$$\int T(w,\mu,s,\cdot)\,\mu(ds) = \mu(\cdot), \tag{2.8}$$

holds for all $(w,\mu) \in A$. Then we can redefine T on $((W \times M)\setminus A) \times G$ by $T(w,\mu,s,\cdot) := \delta_s$, to obtain a kernel T satisfying (2.8) for all $(w,\mu) \in W \times M$. If A is *invariant* (i.e. $\theta_s A = A$, $s \in G$) and T is invariant, then the modified T is an invariant kernel too. A similar remark applies to allocation rules.

By Theorem 7.2 in [6] (X,ξ) is mass-stationary, iff

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t(X,\xi) \in A\}\,T(X,\xi,0,dt)\right] = \mathbb{P}((X,\xi) \in A), \quad A \in \mathcal{F}', \tag{2.9}$$

holds for all invariant mass-preserving weighted transport-kernels T.

A measure $\mu \in M$ is *discrete* if

$$\mu = \sum_{s:\mu\{s\}>0} \mu\{s\}\delta_s$$

and *diffuse* if $\mu\{s\} = 0$ for all $s \in G$. Lemma 2.2 in [3] shows that any $\mu \in M$ can be measurably and uniquely written as the sum of a discrete measure μ^d and a diffuse measure μ^c. The proof of this result shows that the mapping $\mu \mapsto (\mu^d, \mu^c)$ is covariant in the obvious sense. Therefore the characterization (2.5) of mass-stationarity together with $\xi = \xi^d + \xi^c$ implies the following result.

Proposition 2.4. *If (X, ξ^d) and (X, ξ^c) are both mass-stationary, then (X, ξ) is mass-stationary.*

3 Bernoulli transports

A *Bernoulli transport-kernel* is a transport-kernel T of the form

$$T(w, \mu, s, \cdot) = p(w, \mu, s)\delta_s + (1 - p(w, \mu, s))\delta_{\tau(w,\mu,s)}, \quad (w, \mu, s) \in W \times M \times G, \tag{3.1}$$

where $p : W \times M \times G \to [0, 1]$ is measurable and $\tau : W \times M \times G \to G$ is a measurable mapping. Invariance of Bernoulli transport-kernels can easily be characterized as follows.

Lemma 3.1. *Let T be a Bernoulli transport-kernel as in* (3.1) *such that for all $(w, \mu) \in W \times M$ it holds that $p(w, \mu, s) = 1$ iff $\tau(w, \mu, s) = s$. Then T is invariant iff τ is covariant and $p(w, \mu, s) = p(\theta_s(w, \mu), 0)$ for all $(w, \mu, s) \in W \times M \times G$.*

Recall that (X, ξ) is a random pair such that $\mathbb{P}((X, \xi) \in \cdot)$ is σ-finite and $\mathbb{P}(\xi(G) = 0) = 0$. We will show that the validity of (2.9) for all invariant Bernoulli transport-kernels is sufficient for mass-stationarity of (X, ξ). The *support* of a measure $\mu \in M$ is denoted by $\operatorname{supp} \mu$. Here we need to make the weak assumption, that $(W, \mathcal{W})$ is a *Borel space*, i.e. Borel isomorphic to a Borel subset of $[0, 1]$, see e.g. Appendix A1 in [4].

Theorem 3.2. *Assume that $(W, \mathcal{W})$ is a Borel space, that $\mathbb{P}(0 \notin \operatorname{supp} \xi) = 0$, and that $\mathbb{P}(\xi \neq \xi^d) = 0$. Assume also that* (2.9) *holds for all invariant mass-preserving Bernoulli transport-kernels T. Then (X, ξ) is mass-stationary.*

Our proof of Theorem 3.2 requires the following generalization of a result in [2]. A proof can be found in [5]. A *matching* is an allocation rule τ such that the following holds for all $(w, \mu) \in W \times M$: $\tau(w, \mu, s) \in \operatorname{supp} \mu$ and $\tau(w, \mu, \tau(w, \mu, s)) = s$ for all $s \in \operatorname{supp} \mu$, and $\tau(w, \mu, s) = s$ for all $s \notin \operatorname{supp} \mu$.

Lemma 3.3. *Assume that $(W, \mathcal{W})$ is a Borel space. Then there exist invariant matchings τ_k, $k \in \mathbb{N}$, such that for all $(w, \mu) \in W \times M$ with $\operatorname{supp} \mu$ locally finite and $0 \in \operatorname{supp} \mu$*

$$\{0\} \cup \{t \in \operatorname{supp} \mu : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \tag{3.2}$$

For $n \in \mathbb{N}$ and $\mu \in M$ we define $\mu_n \in M$ by

$$\mu_n(B) := \int_B \mathbf{1}\{1/n \le \mu\{s\} \le n\}\, \mu(ds), \quad B \in \mathcal{G}.$$

Then $1/n \le \mu_n\{s\} \le n$, $s \in \operatorname{supp} \mu_n$, and $\operatorname{supp} \mu_n$ is locally finite. We will use the following version of Lemma 3.3.

Lemma 3.4. *Assume that $(W, \mathcal{W})$ is a Borel space and let $n \in \mathbb{N}$. Then there exist invariant matchings τ_k, $k \in \mathbb{N}$, such that for all $(w, \mu) \in W \times M$ with $0 \in \operatorname{supp} \mu_n$*

$$\{0\} \cup \{t \in \operatorname{supp} \mu_n : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \tag{3.3}$$

Furthermore, the τ_k can be chosen such that the following holds for all $(w, \mu) \in W \times M$. If $s \notin \operatorname{supp} \mu_n$ then $\tau_k(w, \mu, s) = s$ and if $s \in \operatorname{supp} \mu_n$ then $\tau_k(w, \mu, s) \in \operatorname{supp} \mu_n$.

Proof. We apply Lemma 3.3 with W replaced by $W \times M$. This gives matchings τ'_k, $k \in \mathbb{N}$, such that for all $(w, \mu, \nu) \in W \times M \times M$ with $\operatorname{supp} \nu$ locally finite and $0 \in \operatorname{supp} \nu$

$$\{0\} \cup \{t \in \operatorname{supp} \nu : \theta_t(w, \mu, \nu) \neq (w, \mu, \nu)\} \subset \{\tau'_k((w, \mu), \nu, 0) : k \in \mathbb{N}\}.$$

For any $k \in \mathbb{N}$ we define a mapping $\tau_k : W \times M \times G \to G$ by $\tau_k(w, \mu) := \tau'_k((w, \mu), \mu_n)$. Then (3.3) holds. (Note that $\theta_t(w, \mu, \mu_n) = (w, \mu, \mu_n)$ iff $\theta_t(w, \mu) = (w, \mu)$.) It is now easy to see that the τ_k are invariant matchings with the properties stated in the lemma. $\square$

Proof of Theorem 3.2. It is convenient (and no restriction of generality) to assume that $(\Omega, \mathcal{F}) = (W \times M, \mathcal{W} \otimes \mathcal{M})$, $\mathbb{P} = \mathbb{P}((X, \xi) \in \cdot)$, and that (X, ξ) is the identity on $W \times M$. We will prove the Mecke equation (2.5). Satz 2.5 in [7] (see also Section 2 in [6]) shows that $\mathbb{P}$ is the Palm measure of (X, ξ) w.r.t. a σ-finite invariant measure on Ω. By Theorem 7.3 in [6] this is equivalent to mass-stationarity of (X, ξ).

In the sequel we fix $n \in \mathbb{N}$. Let τ be an invariant matching with the properties listed after (3.3). Define a Bernoulli transport-kernel T by

$$T(w, \mu, s, \cdot) := \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_s + \frac{\mu\{\tau(s)\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_{\tau(s)} \tag{3.4}$$

if $s \in \operatorname{supp} \mu_n$, and $T(w, \mu, s, \cdot) := \delta_s$, otherwise. Here and below we skip the argument (w, μ) whenever possible. This transport-kernel is of the form (3.1) with

$$p(s) := \mathbf{1}\{\tau(s) \neq s\} \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} + \mathbf{1}\{\tau(s) = s\}, \tag{3.5}$$

where we recall that $\tau(s) = s$ for $s \notin \operatorname{supp} \mu_n$. We have

$$p(\theta_s, 0) = \mathbf{1}\{\tau(\theta_s, 0) \neq 0\} \frac{\theta_s \mu\{0\}}{\theta_s \mu\{0\} + \theta_s \mu\{\tau(\theta_s, 0)\}} + \mathbf{1}\{\tau(\theta_s, 0) = 0\}.$$

Since $\tau(\theta_s, 0) = \tau(s) - s$ and $\theta_s \mu\{t\} = \mu\{t + s\}$, $t \in G$, we obtain that $p(\theta_s, 0) = p(s)$. Lemma 3.1 implies that T is invariant.

We next prove that T is mass-preserving, i.e.

$$\int T(w,\mu,s,\{t\})\,\mu(ds)=\mu\{t\},\quad t\in G,\, w\in W,\, \mu\in M. \tag{3.6}$$

Fix $w\in W$ and $\mu\in M$, and take $t\in G$. Assume first that $t\notin \operatorname{supp}\mu_n$. Then $\tau(t)=t$ and $T(t,\{t\})=1$. Let $s\in G\setminus\{t\}$. If $s\notin\operatorname{supp}\mu_n$ then $\tau(s)=s$ and $T(s,\{t\})=0$. If $s\in\operatorname{supp}\mu_n$, then $T(s,\{t\})>0$ is only possible if $\tau(s)=t$, i.e. $\tau(t)=s$. As this would contradict $\tau(t)=t$, we again get $T(s,\{t\})=0$. Hence $T(s,\{t\})=\mathbf{1}\{s=t\}$, implying (3.6) for $t\notin\operatorname{supp}\mu_n$.

Assume now that $t\in\operatorname{supp}\mu_n$. Then $T(s,\{t\})=0$ for $s\notin\operatorname{supp}\mu_n$. (Otherwise we would obtain that $\tau(s)=t\neq s$.) For $s\in\operatorname{supp}\mu_n$ we can have $T(s,\{t\})>0$ only if $s=t$ or $\tau(s)=t$. The latter equality implies $\tau(t)=s$. If $\tau(t)=t$ then $T(s,\{t\})=0$ for all $s\in\operatorname{supp}\mu_n\setminus\{t\}$ and thus (3.6) holds. The only non-trivial case is $\tau(t)\neq t$. Then the left-hand side of (3.6) equals

$$\begin{aligned}
&\mu\{t\}T(t,\{t\})+\mu\{\tau(t)\}T(\tau(t),\{t\})\\
&\quad=\mu\{t\}\frac{\mu\{t\}}{\mu\{t\}+\mu\{\tau(t)\}}+\mu\{\tau(t)\}\frac{\mu\{t\}}{\mu\{t\}+\mu\{\tau(t)\}}=\mu\{t\},
\end{aligned}$$

where we have again used that $\tau(\tau(t))=t$.

We have established that T is an invariant mass-preserving Bernoulli transport-kernel and will now head towards (2.5). Let us define the *mass-shift* $\theta_\tau:\Omega\to\Omega$ by $\theta_\tau(\omega):=\theta_{\tau(\omega,0)}(\omega)$. (We also define the random measure $\theta_\tau\xi$ by $\theta_\tau\xi(\omega):=\theta_{\tau(\omega,0)}\xi(\omega)$; the random measure $\theta_\tau\xi_n=(\theta_\tau\xi)_n$ is defined in the same way.) A quick consequence of the matching property of τ is

$$\tau(\theta_\tau,0)=-\tau(0). \tag{3.7}$$

In particular we have

$$\mathbf{1}_A(\theta_\tau)=\mathbf{1}_A, \tag{3.8}$$

where $A:=\{\tau(0)\neq 0\}$. Note that $A\subset\{0\in\operatorname{supp}\xi_n,\tau(0)\in\operatorname{supp}\xi_n\}$. Let $f:\Omega\to[0,\infty)$ be measurable with $\mathbb{E}_{\mathbb{P}}[f]<\infty$. Let $B\in\mathcal{G}$ and define $g(\omega,s):=f(\omega)\mathbf{1}\{s\in B\}$. By assumption and the facts established above we can apply (2.9) for our specific T, to obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}[\mathbf{1}_A g(\theta_0,\tau(0))\xi\{\tau(0)\}]&=\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(\theta_s)g(\theta_s,\tau(\theta_s,0))\xi(\theta_s,\{\tau(\theta_s,0)\}\,T(0,ds)\right]\\
&=\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_A g(\theta_0,\tau(0))\xi\{\tau(0)\}p(0)\right]+\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_A g(\theta_\tau,-\tau(0))\xi(\theta_\tau,\{-\tau(0)\})(1-p(0))\right],
\end{aligned}$$

where we have used (3.8) and (3.7) for the second equality. (We suppress the dependence on (X,ξ) in the notation; for instance we use θ_s as a shorthand for $\theta_s(X,\xi)$.) Recalling the definition of p and using $\theta_\tau\xi\{-\tau(0)\}=\xi\{0\}$, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}[\mathbf{1}_A g(\theta_0,\tau(0))\xi\{\tau(0)\}]\\
&\quad=\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_A g(\theta_0,\tau(0))\frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\}+\xi\{\tau(0)\}}\right]+\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_A g(\theta_\tau,-\tau(0))\frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\}+\xi\{\tau(0)\}}\right].
\end{aligned}$$

Since for $0 \in \operatorname{supp} \xi_n$ and $\tau(0) \in \operatorname{supp} \xi_n$

$$g(\theta_0, \tau(0)) \frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \leq f \frac{n^3}{2}, \quad g(\theta_0, \tau(0))\xi\{\tau(0)\} \leq fn,$$

and $\mathbb{E}_{\mathbb{P}}[f] < \infty$, we get by subtraction

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_A g(\theta_0, \tau(0)) \frac{\xi\{\tau(0)\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \right] = \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_A g(\theta_\tau, -\tau(0)) \frac{\xi\{\tau(0)\}\xi\{0\}}{\xi\{0\} + \xi\{\tau(0)\}} \right]. \tag{3.9}$$

Consider the function $\tilde{g} : \Omega \times G \to [0, \infty)$ given by

$$\tilde{g}(s) := \mathbf{1}\{0 \in \operatorname{supp} \xi_n, s \in \operatorname{supp} \xi_n\} \frac{\xi\{0\} + \xi\{s\}}{\xi\{s\}}.$$

We have

$$\begin{aligned} \tilde{g}(\theta_\tau, -\tau(0)) &= \mathbf{1}\{0 \in \operatorname{supp} \theta_\tau \xi_n, -\tau(0) \in \operatorname{supp} \theta_\tau \xi_n\} \frac{\theta_\tau \xi\{0\} + \theta_\tau \xi\{-\tau(0)\}}{\theta_\tau \xi\{-\tau(0)\}} \\ &= \mathbf{1}\{\tau(0) \in \operatorname{supp} \xi_n, 0 \in \operatorname{supp} \xi_n\} \frac{\xi\{\tau(0)\} + \xi\{0\}}{\xi\{0\}}. \end{aligned}$$

Since $\tilde{g}(\theta_0, \tau(0)) \leq 2n^2$ and $\tilde{g}(\theta_\tau, -\tau(0)) \leq 2n^2$, we can apply (3.9) with $g \cdot \tilde{g}$ instead of g. Together with monotone convergence this gives for all measurable $g : \Omega \times G \to [0, \infty)$:

$$\mathbb{E}_{\mathbb{P}} \left[\mathbf{1}\{\tau(0) \neq 0\} g(\theta_0, \tau(0)) \xi_n\{\tau(0)\} \right] = \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}\{\tau(0) \neq 0\} g(\theta_\tau, -\tau(0)) \xi_n\{\tau(0)\} \right]. \tag{3.10}$$

We now apply Lemma 3.4. If $0 \in \operatorname{supp} \xi_n$, then (3.3) yields that

$$\int h(t) \mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\} \, \xi_n(dt) = \sum_{k \in \mathbb{N}} h_k(X, \xi, \tau_k(0)) h(\tau_k(0)) \xi_n\{\tau_k(0)\} \tag{3.11}$$

for all measurable $h : W \times G \to [0, \infty)$, where

$$h_k(t) := \mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\} \mathbf{1}\{\tau_l(0) \neq t \text{ for } 1 \leq l \leq k-1\}.$$

We claim that

$$h_k(\theta_{\tau_k}(X, \xi), -\tau_k(0)) = h_k(X, \xi, \tau_k(0)), \quad k \in \mathbb{N}. \tag{3.12}$$

Indeed, for $k \geq 2$ and $l \leq k-1$ we have by covariance of τ_l that $\tau_l(\theta_{\tau_k}, 0) = -\tau_k(0)$ iff $\tau_l(\tau_k(0)) = 0$. By the matching property of τ_l this is in turn equivalent to $\tau_k(0) = \tau_l(0)$. From (3.11), (3.10) and (3.12) we obtain

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\int \mathbf{1}\{0 \in \operatorname{supp} \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(X, \xi, t) \, \xi_n(dt) \right] \\ &\qquad = \sum_{k \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[h_k(X, \xi, \tau_k(0)) g(X, \xi, \tau_k(0)) \xi_n\{\tau_k(0)\} \right] \\ &\qquad = \sum_{k \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[h_k(X, \xi, \tau_k(0)) g(\theta_{\tau(k)}(X, \xi), -\tau_k(0)) \xi_n\{\tau_k(0)\} \right]. \end{aligned}$$

Using (3.12) again we arrive at

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{0 \in \operatorname{supp}\xi_n, \theta_t(X,\xi) \neq (X,\xi)\} g(X,\xi,t)\,\xi_n(dt)\right]$$

$$= \mathbb{E}_{\mathbb{P}} \int \mathbf{1}\{0 \in \operatorname{supp}\xi_n, \theta_t(X,\xi) \neq (X,\xi)\} g(\theta_t(X,\xi),-t)\,\xi_n(dt). \quad (3.13)$$

Let $t \in \operatorname{supp}\xi_n$ be such that $\theta_t(X,\xi) = (X,\xi)$. Then $\xi_n = \theta_{-t}\xi_n$ and

$$\xi_n\{t\} = \theta_t\xi_n\{0\} = \theta_{-t}\xi_n\{0\} = \xi\{-t\}.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{0 \in \operatorname{supp}\xi_n, \theta_t(X,\xi) = (X,\xi)\} g(X,\xi,t)\,\xi_n(dt)\right]$$

$$= \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{0 \in \operatorname{supp}\xi_n, \theta_t(X,\xi) = (X,\xi)\} g(\theta_t(X,\xi),-t)\,\xi_n(dt)\right].$$

Adding this to (3.13) and taking the limit as $n \to \infty$, yields (2.5) and hence the assertion of the theorem. ☐

Remark 3.5. The last part of the preceding proof (starting with (3.12)) coincides with the second half of the proof of Theorem 1.1 in [2]. But it does also close a gap in the latter proof in that it is using Lemma 3.3 instead of the (slightly) weaker Theorem 3.6 in [2]. This theorem is not sufficient for the conclusion made in [2].

The definitions of the previous section apply in particular in the case where W is a singleton. In this case we can identify $W \times M$ with M and abbreviate the set of all mass-preserving invariant weighted transport-kernels as $\mathbf{T}$ and the set of all mass-preserving allocation rules as $\mathbf{A}$. Moreover, the set of all mass-preserving invariant Bernoulli transport-kernels (a subset of $\mathbf{T}$) is denoted by $\mathbf{T}_b$, while the set of all invariant matchings (a subset of $\mathbf{A}$) is denoted by $\mathbf{A}_m$.

The proof of Theorem 3.2 yields the following result without a Borel assumption on the space W.

Proposition 3.6. *Assume that $\mathbb{P}(0 \notin \operatorname{supp}\xi) = 0$ and $\mathbb{P}(\xi \neq \xi^d) = 0$. Assume further that*

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t(X,\xi) \in A\}\, T(\xi,0,dt)\right] = \mathbb{P}((X,\xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (3.14)$$

holds for all $T \in \mathbf{T}_b$. Then, for all measurable $g : W \times M \times G \to [0,\infty)$,

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t\xi \neq \xi\} g(\theta_t(X,\xi),-t)\,\xi(dt)\right] = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t\xi \neq \xi\} g(X,\xi,t)\,\xi(dt)\right]. \quad (3.15)$$

Proof: Let $n \in \mathbb{N}$. We apply Lemma 3.4 in the case where W is a singleton. We can then proceed as in the proof of Theorem 3.2, to obtain as at (3.13)

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t\xi \neq \xi\} g(X,\xi,t)\,\xi_n'(dt)\right] = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_t\xi \neq \xi\} g(\theta_t(X,\xi),-t)\,\xi_n'(dt)\right].$$

where $\xi'_n(dt) = \mathbf{1}\{0 \in \operatorname{supp} \xi_n\}\xi_n(dt)$. Letting $n \to \infty$ gives the assertion. $\square$

Let $N \subset M$ be the set of all discrete measures μ on G having $\mu\{s\} \in \{0, 1\}$ for all $s \in G$. Strengthening the assumptions of Proposition 3.6, we can use a simplified version of the proof of Theorem 3.2 to get the following result. We refer here also to Theorem 1.1 in [2].

Proposition 3.7. *Assume* $\mathbb{P}(0 \notin \operatorname{supp} \xi) = 0$, $\mathbb{P}(\xi \notin N) = 0$, *and that*

$$\mathbb{P}(\theta_\tau(X, \xi) \in A) = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \tag{3.16}$$

holds for all $\tau \in \mathbf{A}_m$, *where* $\theta_\tau : W \times M \to \Omega$ *is defined by* $\theta_\tau(w, \mu) := \theta_{\tau(\mu,0)}(w, \mu)$. *Then* (3.15) *holds for all measurable* $g : W \times M \times G \to [0, \infty)$.

A measure $\mu \in M$ is called *periodic* if $\theta_t\mu = \mu$ for some $t \neq 0$. A measure $\mathbb{Q}$ on M is called *aperiodic* if it is supported by the set of all measures $\mu \in M$ that are not periodic. Since the Mecke equation (2.5) implies mass-stationarity, Propositions 3.6 and 3.7 give the following result.

Proposition 3.8. *Assume that* $\mathbb{P}(0 \notin \operatorname{supp} \xi) = 0$ *and* $\mathbb{P}(\xi \neq \xi^d) = 0$. *Assume further that* $\mathbb{P}(\xi \in \cdot)$ *is aperiodic. If either* (3.14) *holds for all* $T \in \mathbf{T}_b$ *or* $\mathbb{P}(\xi \notin N) = 0$ *and* (3.16) *holds for all* $\tau \in \mathbf{A}_m$, *then* (X, ξ) *is mass-stationary.*

Remark 3.9. Assume that $\mathbb{P}(0 \notin \operatorname{supp} \xi) = 0$ and $\mathbb{P}(\xi \neq \xi^d) = 0$. If (3.14) holds for all $T \in \mathbf{T}_b$ we conjecture that (X, ξ) is mass-stationary without the additional aperiodicity assumption.

Remark 3.10. Let $\mathbb{P}$ satisfy the assumptions of Proposition 3.8 and assume in addition that $\mathbb{P}(\xi \notin N) = 0$. If $\mathbb{P}(\xi \in \cdot)$ is not aperiodic, then Proposition 3.8 does not apply. However, we might assume that (3.16) holds for all $\tau \in \mathbf{A}$. We believe that this implies mass-stationarity of (X, ξ). In case $G = \mathbb{R}^d$ this was established in Theorem 4.1 in [1].

Remark 3.11. Let the assumptions of Proposition 3.7 be satisfied. Example 7.1 in [6] shows that invariance of $\mathbb{P}((X, \xi) \in \cdot)$ under mass-preserving allocation rules (in the sense of (3.16)) is not enough to imply mass-stationarity of (X, ξ). Therefore Theorem 3.2 does not only solve Problem 7.3 in [6] for discrete random measures (up to the fact that in case of periodicities we have to allow the weighted transport-kernels to depend on X) but is also the natural (and minimal) extension of Theorem 1.1 in [2] to discrete random measures.

4 Cox transports

For any $\alpha \in M$ we let Π_α denote the distribution of a Poisson process with intensity measure α. It is convenient to consider Π_α as a probability measure on M. It is concentrated on those $\mu \in M$ having locally finite support and $\mu\{s\} \in \mathbb{N}_0$, $s \in G$. We consider a *Cox process* (see e.g. [4]) driven by (X, ξ), i.e. a random measure ζ on G satisfying

$$\mathbb{P}((X, \xi, \zeta) \in \cdot) = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{(X, \xi, \mu) \in \cdot\}\, \Pi_\xi(d\mu)\right]. \tag{4.1}$$

Possibly extending $(\Omega, \mathcal{F}, \mathbb{P})$, the existence of ζ can be assumed without loss of generality. Let $\zeta^0 := \zeta + \delta_0$ and define $\Pi^0_\alpha := \int \mathbf{1}\{\mu + \delta_0 \in \cdot\}\, \Pi_\alpha(d\mu)$, $\alpha \in M$.

Theorem 4.1. *Assume that $\mathbb{P}(X \in \cdot)$ is σ-finite. Then (X, ξ) is mass-stationary iff (X, ζ^0) is mass-stationary. In this case even $((X, \xi), \zeta^0)$ is mass-stationary.*

We will prove this theorem later in this section.

Remark 4.2. Assume that $\mathbb{P}(X \in \cdot)$ is σ-finite and that (X, ξ) is mass-stationary. Then Theorem 4.1 and (2.9) imply

$$\mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}_A(\theta_s X, \theta_s \xi, \theta_s \mu)\, T(X, \xi, \mu, 0, ds)\, \Pi^0_\xi(d\mu)\right] = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(X, \xi, \mu)\, \Pi^0_\xi(d\mu)\right] \quad (4.2)$$

for all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ and all mass-preserving invariant weighted transport-kernels T from $(W \times M) \times M \times G$ to G.

Combining Theorem 4.1 with Proposition 3.8 gives the following characterization of mass-stationarity via Bernoulli transport-kernels. Recall the definitions of the sets $\mathbf{T}$, $\mathbf{T}_b$, $\mathbf{A}$, and $\mathbf{A}_m$ given before Remark 2.3.

Corollary 4.3. *Assume that $\mathbb{P}(X \in \cdot)$ is σ-finite. Then (X, ξ) is mass-stationary iff*

$$\mathbb{E}_{\mathbb{P}}\left[\iint \mathbf{1}_A(\theta_s X, \theta_s \mu)\, T(\mu, 0, ds)\, \Pi^0_\xi(d\mu)\right] = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(X, \mu)\, \Pi^0_\xi(d\mu)\right] \quad (4.3)$$

holds for all $A \in \mathcal{W} \otimes \mathcal{M}$ and all $T \in \mathbf{T}_b$.

Proof: If (X, ξ) is mass-stationary then (4.3) follows as a special case of (4.2). Conversely, assume that (4.3) holds. The properties of a Poisson process imply

$$\Pi^0_\alpha(\{\mu \in M : \theta_s \mu = \mu \text{ for some } s \in \operatorname{supp} \mu \setminus \{0\}\}) = 0, \quad \alpha \in M. \quad (4.4)$$

It follows that $\mathbb{P}(\zeta^0 \in \cdot)$ is aperiodic. Hence we obtain from Proposition 3.8 that (X, ζ^0) is mass-stationary. Theorem 4.1 yields mass-stationarity of (X, ξ). □

For diffuse random measures the condition (4.3) can be simplified as follows.

Corollary 4.4. *Assume that $\mathbb{P}(X \in \cdot)$ is σ-finite and that $\mathbb{P}(\xi \neq \xi^c) = 0$. Then (X, ξ) is mass-stationary iff*

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(\theta_{\tau(\mu,0)} X, \theta_{\tau(\mu,0)} \mu)\, \Pi^0_\xi(d\mu)\right] = \mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(X, \mu)\, \Pi^0_\xi(d\mu)\right], \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (4.5)$$

holds for all $\tau \in \mathbf{A}_m$.

Proof: Using the second part of Proposition 3.8, the result can be proved as Corollary 4.3. □

Remark 4.5. Equation (4.3) can be written as

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}_A(\theta_s X, \theta_s \zeta^0)\, T(\zeta^0, 0, ds)\right] = \mathbb{P}((X, \zeta^0) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}. \quad (4.6)$$

The point here is that the random measure ξ is not entering this equation explicitly, but only implicitly, as random intensity measure of ζ.

Remark 4.6. Assume that $\mathbb{P}(X \in \cdot)$ is σ-finite. Let T be a transport-kernel. Define another transport-kernel T' by

$$T'(w, \alpha, s, \cdot) := \int T(w, \mu + \delta_s, s, \cdot)\, \Pi_\alpha(d\mu). \tag{4.7}$$

Then (4.8) below implies invariance of T', while (4.9) easily implies that T' is mass-preserving. If (X, ξ) is mass-stationary, then Remark 4.2 yields

$$\mathbb{E}_{\mathbb{P}}\left[\int \mathbf{1}\{\theta_s(X, \xi) \in \cdot\}\, T'(X, \xi, 0, ds)\right] = \mathbb{P}((X, \xi) \in \cdot).$$

We do not know whether the validity of this equation for all such *Cox transport-kernels* T' is enough to imply mass-stationarity of (X, ξ). We refer here also to Problem 7.3 in [6].

Remark 4.7. Take $\tau \in \mathbf{A}$, and let $V := \tau(\zeta^0, 0)$. Then (4.5) can be written as

$$(\theta_V X, \theta_V \zeta^0) \overset{d}{=} (X, \zeta^0).$$

Remark 4.8. Assuming that $\mathbb{P}(\xi \in \cdot)$ is σ-finite is stronger than only assuming that $\mathbb{P}((X, \xi) \in \cdot)$ is σ-finite. If, for instance, X is a constant, $\mathbb{P}(X \in \cdot)$ can only be σ-finite, if $\mathbb{P}$ is a finite measure. We do not know, whether the results of this section remain true in the more general case, where only $\mathbb{P}((X, \xi) \in \cdot)$ is σ-finite.

Proof of Theorem 4.1: First we recall that

$$\int \mathbf{1}\{\mu \in \cdot\}\, \Pi_{\theta_s \alpha}(d\mu) = \int \mathbf{1}\{\theta_s \mu \in \cdot\}\, \Pi_\alpha(d\mu), \quad \alpha \in M,\ s \in G, \tag{4.8}$$

and

$$\iint \mathbf{1}\{(\mu, s) \in \cdot\}\, \mu(ds)\, \Pi_\alpha(d\mu) = \iint \mathbf{1}\{(\mu + \delta_s, s) \in \cdot\}\, \alpha(ds)\, \Pi_\alpha(d\mu), \quad \alpha \in M. \tag{4.9}$$

The first equation comes directly from the definition of Π_α, while the second is from [7].

Assume now that (X, ξ) is mass-stationary. By Theorem 6.3 in [6] there is a stationary σ-finite measure $\mathbb{Q}$ on $W \times M$ such that

$$\mathbb{P}((X, \xi) \in \cdot) = \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(w, \mu)) \mathbf{1}_B(s)\, \mu(ds)\, \mathbb{Q}(d(w, \mu)), \quad A \in \mathcal{W} \otimes \mathcal{M}, \tag{4.10}$$

where $0 < \lambda(B) < \infty$. This means that $\mathbb{P}((X, \xi) \in \cdot)$ is the Palm measure of the projection from $W \times M$ onto M with respect to $\mathbb{Q}$, cf. (2.2).

Consider the measurable space $(\Omega^*, \mathcal{F}^*) := (\Omega \times M \times M, \mathcal{F} \otimes \mathcal{M} \otimes \mathcal{M})$ equipped with the measurable flow $\theta^*_s(w, \alpha, \mu) := (\theta_s w, \theta_s \alpha, \theta_s \mu)$. Define a measure $\mathbb{Q}^*$ on $(\Omega^*, \mathcal{F}^*)$ by

$$\mathbb{Q}^* := \iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\}\, \Pi_\alpha(d\mu)\, \mathbb{Q}(d(w, \alpha)). \tag{4.11}$$

Since $\mathbb{Q}$ is σ-finite, so is $\mathbb{Q}^*$. Using (4.8), we get for any measurable $f : \Omega^* \to [0, \infty)$

$$\int f(\theta_s^*(w, \alpha, \mu))\, \mathbb{Q}^*(d(w, \alpha, \mu)) = \iint f(\theta_s w, \theta_s \alpha, \mu)\, \Pi_{\theta_s \alpha}(d\mu)\, \mathbb{Q}(d(w, \alpha))$$
$$= \iint f(w, \alpha, \mu)\, \Pi_\alpha(d\mu)\, \mathbb{Q}(d(w, \alpha)),$$

where the second equality comes from stationarity of $\mathbb{Q}$. Hence $\mathbb{Q}^*$ is invariant under the flow $\{\theta_s^* : s \in G\}$.

Denote by (X^*, ξ^*, ζ^*) the identity on Ω^*. Our next aim is to compute the Palm measure of $((X^*, \xi^*), \zeta^*)$ w.r.t. $\mathbb{Q}^*$. Using (4.8) and (4.9), we obtain for all measurable $f : \Omega^* \times G \to [0, \infty)$ that

$$\iint f(\theta_s(w, \alpha), \theta_s \mu, s)\, \mu(ds)\, \mathbb{Q}^*(d(w, \alpha, \mu))$$
$$= \iiint f(\theta_s(w, \alpha), \theta_s \mu, s)\, \mu(ds)\, \Pi_\alpha(d\mu)\, \mathbb{Q}(d(w, \alpha))$$
$$= \iiint f(\theta_s(w, \alpha), \theta_s(\mu + \delta_s), s)\, \alpha(ds)\, \Pi_\alpha(d\mu)\, \mathbb{Q}(d(w, \alpha))$$
$$= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s)\, \Pi_{\theta_s \alpha}(d\mu)\, \alpha(ds)\, \mathbb{Q}(d(w, \alpha))$$
$$= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s)\, \Pi_\alpha(d\mu)\, ds\, \mathbb{P}((X, \xi) \in d(w, \alpha)),$$

where the final equality is due to (4.10) and the refined Campbell theorem (2.3) for the pair $(\mathbb{Q}, \mathbb{P}((X, \xi) \in \cdot))$. Therefore

$$\iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\}\, \Pi_\alpha^0(d\mu)\, \mathbb{P}((X, \xi) \in d(w, \alpha)) = \mathbb{P}((X, \xi, \zeta^0) \in \cdot) \tag{4.12}$$

is the Palm measure of $((X^*, \xi^*), \zeta^*)$ w.r.t. $\mathbb{Q}^*$. Theorem 6.3 in [6] implies that $((X, \xi), \zeta^0)$ is mass-stationary and that

$$\mathbb{E}_\mathbb{P}\left[\int g(\theta_s(X, \xi), \theta_s \zeta^0, -s)\, \zeta^0(ds)\right] = \mathbb{E}_\mathbb{P}\left[\int g(X, \xi, \zeta^0, s)\, \zeta^0(ds)\right] \tag{4.13}$$

for any measurable $g : W \times M \times M \times G \to [0, \infty)$. In particular we have

$$\mathbb{E}_\mathbb{P}\left[\int g(\theta_s(X, \zeta^0), -s)\, \zeta^0(ds)\right] = \mathbb{E}_\mathbb{P}\left[\int g(X, \zeta^0, s)\, \zeta^0(ds)\right] \tag{4.14}$$

for any measurable $g : W \times M \times G \to [0, \infty)$. As σ-finiteness of $\mathbb{P}(X \in \cdot)$ entails the same property of $\mathbb{P}((X, \zeta^0) \in \cdot)$, we conclude that (X, ζ^0) is mass-stationary.

To prove the other implication, we assume that (X, ζ^0) is mass-stationary. Since mass-stationarity is equivalent to the Mecke equation (4.14), we have

$$\mathbb{E}_\mathbb{P}\left[\iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s)\, (\mu + \delta_0)(ds)\, \Pi_\xi(d\mu)\right]$$
$$= \mathbb{E}_\mathbb{P}\left[\iint f(X, \mu + \delta_0, s)\, (\mu + \delta_0)(ds))\, \Pi_\xi(d\mu)\right]$$

for all measurable $f : W \times M \times G \to [0, \infty)$. If $\mathbb{E}_{\mathbb{P}}[\int f(X, \mu + \delta_0, 0)\, \Pi_\xi(d\mu)] < \infty$, we obtain

$$\mathbb{E}_{\mathbb{P}}\left[\iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s)\, \mu(ds)\, \Pi_\xi(d\mu)\right] = \mathbb{E}_{\mathbb{P}}\left[\iint f(X, \mu + \delta_0, s)\, \mu(ds)\, \Pi_\xi(d\mu)\right].$$

Since $\mathbb{P}(X \in \cdot)$ is σ-finite, this remains true for any measurable $f : W \times M \times G \to [0, \infty)$. Using (4.9) and then (4.8) we get

$$\mathbb{E}_{\mathbb{P}}\left[\iint f(\theta_s X, \mu + \delta_{-s} + \delta_0, -s)\, \Pi_{\theta_s \xi}(d\mu)\, \xi(ds)\right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[\iint f(X, \mu + \delta_s + \delta_0, s)\, \Pi_\xi(d\mu)\, \xi(ds)\right].$$

We apply this with $f(w, \mu, s) := \mathbf{1}\{\mu\{s\} \geq 1, \mu\{0\} \geq 1\} f_1(w, \mu - \delta_s - \delta_0, s)$ for a measurable function $f_1 : W \times M \times G \to [0, \infty)$. It follows that

$$\mathbb{E}_{\mathbb{P}}\left[\iint f_1(\theta_s X, \mu, -s)\, \Pi_{\theta_s \xi}(d\mu)\, \xi(ds)\right] = \mathbb{E}_{\mathbb{P}}\left[\iint f_1(X, \mu, s)\, \Pi_\xi(d\mu)\, \xi(ds)\right]. \tag{4.15}$$

Take $B \in \mathcal{G}$ and measurable functions $h_1 : W \to \mathbb{R}$ and $h : M \to \mathbb{R}$. Equation (4.15) implies

$$\mathbb{E}_{\mathbb{P}}\left[\int h_1(\theta_s X) h^*(\theta_s \xi) \mathbf{1}_B(-s)\, \xi(ds)\right] = \mathbb{E}_{\mathbb{P}}[h_1(X) \xi(B) h^*(\xi)], \tag{4.16}$$

where the measurable function $h^* : M \to [0, \infty]$ is defined by

$$h^*(\alpha) := \int h(\mu) \Pi_\alpha(d\mu). \tag{4.17}$$

Our next aim is to show that the class of measurable functions defined by (4.17) is rich enough, to conclude from (4.16) that

$$\mathbb{E}_{\mathbb{P}}\left[\int h_1(\theta_s X) g(\theta_s \xi) \mathbf{1}_B(-s)\, \xi(ds)\right] = \mathbb{E}_{\mathbb{P}}[h_1(X) \xi(B) g(\xi)] \tag{4.18}$$

holds for all measurable $g : M \to \mathbb{R}$. For $n \in \mathbb{N}$ and $\mu \in M$ we define a measure $\mu^{(n)}$ on G^n by

$$\mu^{(n)}(C) := \int \cdots \int \mathbf{1}_C(s_1, \ldots, s_n)\, \mu_{s_1, \ldots, s_{n-1}}(ds_n) \cdot \ldots \cdot \mu_{s_1}(ds_2)\, \mu(ds_1),$$

where, for $1 \leq k \leq n-1$, the measure $\mu_{s_1, \ldots, s_k}$ on G is defined by

$$\mu_{s_1, \ldots, s_k} := \mathbf{1}\{\mu - \delta_{s_1} - \ldots - \delta_{s_k}(\{s_1, \ldots, s_k\}) \geq 0\}(\mu - \delta_{s_1} - \ldots - \delta_{s_k}).$$

A well-known property of a Poisson process (following from (4.9) and induction) is

$$\int \mu^{(n)}(C)\, \Pi_\alpha(d\mu) = \alpha^n(C), \quad C \in \mathcal{G}^{\otimes n},\ \alpha \in M.$$

For $k, i_1, \dots, i_k \in \mathbb{N}$ and relatively compact sets $B_1, \dots, B_k \in \mathcal{G}$ this gives

$$\int \mu^{(i_1+\ldots+i_k)}(B_1^{i_1} \times \ldots \times B_k^{i_k})\,\Pi_\alpha(d\mu) = \alpha(B_1)^{i_1} \cdot \ldots \cdot \alpha(B_k)^{i_k}. \tag{4.19}$$

Now we consider the measurable function

$$h(\mu) := c_0 + \sum_{i_1,\dots,i_k \in \mathbb{N}} c_{i_1,\dots,i_k} \mu^{(i_1+\ldots+i_k)}(B_1^{i_1} \times \ldots \times B_k^{i_k}), \tag{4.20}$$

where $c_0 \in \mathbb{R}$ and the numbers $c_{i_1,\dots,i_k} \in \mathbb{R}$ satisfy

$$\sum_{i_1,\dots,i_k \in \mathbb{N}} |c_{i_1,\dots,i_k}| x_1^{i_1} \cdot \ldots \cdot x_k^{i_k} < \infty$$

for all $x_1, \dots, x_k \geq 0$. Let the *entire* function $f : \mathbb{R}^k \to \mathbb{R}$ be given by

$$f(x_1, \dots, x_k) := c_0 + \sum_{i_1,\dots,i_k \in \mathbb{N}} c_{i_1,\dots,i_k} x_1^{i_1} \cdot \ldots \cdot x^{i_k}.$$

Then (4.19) and dominated convergence implies that

$$h^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k)), \quad \alpha \in M, \tag{4.21}$$

where we recall the definition (4.17) of h^*. Let $B \in \mathcal{G}$ be relatively compact and $c > 0$. Consider the function $f(x_1, \dots, x_{k+1}) := f(x_1, \dots, x_k)e^{-cx_{k+1}}$, $x_1, \dots, x_{k+1} \in \mathbb{R}$, where f is as in (4.21). Define $\tilde{h}$ as in (4.20) with $(B_1, \dots, B_k)$ replaced by $(B_1, \dots, B_k, B)$ and with the appropriate coefficients $c_{i_1,\dots,i_{k+1}} \in \mathbb{R}$. Then $\tilde{h}^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k))e^{-c\alpha(B)}$ and we get from (4.16) that

$$\mathbb{E}_\mathbb{P}\left[\int h_1(\theta_s X) h(\theta_s \xi) \mathbf{1}_B(-s) e^{-c\xi(B+s)}\,\xi(ds)\right] = \mathbb{E}_\mathbb{P}[h_1(X)h(\xi)\xi(B)e^{-c\xi(B)}] \tag{4.22}$$

holds for all $c > 0$ and all functions h in the class $\mathcal{H}$ of bounded measurable functions of the form (4.21). Assume that $\mathbb{E}_\mathbb{P}[|h_1(X)|] < \infty$. Applying (4.22) with $h \equiv 1$ and h_1 replaced with $|h_1|$, yields

$$\mathbb{E}_\mathbb{P}\left[\int |h_1(\theta_s X)| \mathbf{1}_B(-s) e^{-c\xi(B+s)}\,\xi(ds)\right] = \mathbb{E}_\mathbb{P}[|h_1(X)|\xi(B)e^{-c\xi(B)}] < \infty.$$

Therefore the class of all bounded measurable functions h satisfying (4.22) is a vector space containing the constant functions and being closed under monotone bounded convergence. Since $\mathcal{H}$ is stable under multiplication and generates the σ-field $\mathcal{M}$, we can apply a well-known functional version of the monotone class theorem to obtain that (4.22) holds for any bounded measurable function h. Assume that $h \geq 0$. Since $\mathbb{P}(X \in \cdot)$ is σ-finite, (4.22) remains true for any measurable $h_1 : W \to [0, \infty)$. Moreover, for $c \to 0$ we get from monotone convergence the desired equation (4.18), and in particular

$$\mathbb{E}_\mathbb{P}\left[\int \mathbf{1}_A(\theta_s X, \theta_s \xi, -s)\,\xi(ds)\right] = \mathbb{E}_\mathbb{P}\left[\int \mathbf{1}_A(X, \xi, s)\,\xi(ds)\right], \tag{4.23}$$

for all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ that are of product form. The measure on the right-hand side of (4.23) is finite on product sets of the form $C \times \{\alpha \in M : \alpha(B) \leq k\} \times B$, where $\mathbb{Q}(X \in C) < \infty$, $B \in \mathcal{G}$ is compact, and $k \in \mathbb{N}$. Since $W \times M \times G$ is the monotone union of countably many such sets, (4.23) extends to all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$. This is equivalent to the Mecke equation (2.5) and hence to mass-stationarity of (X, ξ). $\square$

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